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Symmetric positive solutions of fourth order integral BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity on time scales

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ABSTRACT

In this paper, we consider the eigenvalue problems for fourth order integral boundary value problems on time scales for an increasing homeomorphism and homomorphism with sign changing nonlinearities. By using a fixed point index theorem, we give the existence of eigenvalue intervals in which there exist one symmetric positive solution to the problem. An example is also given to demonstrate the main results.

1. Introduction

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that $0, T$ are points in $\mathbb{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is $(0, T) \cap \mathbb{T}$. Throughout this paper, we assume a knowledge of time scales and time scale notation; for more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [1], see the excellent text by Bohner and Peterson [2,3].

The existence of positive solutions for second order nonlinear boundary value problems has been studied by many authors using the fixed point theorem in cones. See [4–7] and references therein.

In this paper, we are concerned with the existence of a symmetric positive solution of the following fourth order boundary value problem with integral boundary conditions

$$(q(t)\phi(p(t)x^{\Delta\nabla}))^{\Delta\nabla} = \lambda f(t, x(t)), \quad t \in (0, 1), \tag{1.1}$$

$$x(0) = x(1) = \int_0^1 g(s)x(s)\nabla s,$$

$$q(0)\phi(p(0)x^{\Delta\nabla}(0)) = q(1)\phi(p(1)x^{\Delta\nabla}(1)) = \int_0^1 h(s)q(s)\phi(p(s)x^{\Delta\nabla}(s))\nabla s, \tag{1.2}$$

where $\lambda > 0$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(i) If $x \le y$, then $\phi(x) \le \phi(y)$, for all $x, y \in \mathbb{R}$;

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(ii) ϕ is continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathbb{R}$.

We will assume that the following assumptions are satisfied:

(H1) $p(t), q(t) : [0, 1] \to (0, \infty)$ are continuous functions, $p(t), q(t)$ are symmetric on $[0, 1]$;
(H2) $f : [0, 1] \times [0, +\infty) \to (-\infty, +\infty)$ is continuous, $f(., x)$ is a symmetric function on $[0, 1]$, $f(t, 0) \geq 0$, $f(1 - t, x) = f(t, x)$ for all $(t, x) \in [0, 1] \times [0, \infty)$;
(H3) $g, h \in \mathcal{C}([0, 1], [0, +\infty))$ are symmetric functions on $[0, 1]$, and $\mu \in [0, 1)$, $\nu \in [0, 1)$, where

$$\mu = \int_0^1 g(s)\nabla s, \qquad \nu = \int_0^1 h(s)\nabla s.$$

The present work is motivated by recent papers [8,9]. In [9], Ma considered the following fourth order boundary value problem with integral boundary conditions

$$u''''(t) = h(t)f(u), \quad 0 < t < 1,$$
$$u(0) = u(1) = \int_0^1 p(s)u(s)\mathrm{d}s,$$
$$u''(0) = u''(1) = \int_0^1 q(s)u(s)\mathrm{d}s,$$

where $p, q \in L^1[0, 1]$, $h : (0, 1) \to [0, +\infty)$ is continuous, symmetric on $(0, 1)$ and may be singular at $t = 0$ and $t = 1$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(., x)$ is symmetric on $[0, 1]$ for all $x \in [0, +\infty)$. The author obtained at least one symmetric positive solution by using the fixed point index in cones.

In [8], Zhang, Feng and Ge studied the existence and nonexistence of symmetric positive solutions of the following fourth order boundary value problem with integral boundary conditions

$$\phi_p(x''(t))'' = w(t)f(t, x(t)), \quad 0 < t < 1,$$
$$x(0) = x(1) = \int_0^1 g(s)x(s)\mathrm{d}s,$$
$$\phi_p(x''(0)) = \phi_p(x''(1)) = \int_0^1 h(s)\phi_p(x''(s))\mathrm{d}s,$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $w \in L^1[0, 1]$ is nonnegative, symmetric on $(0, 1)$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(., x)$ is symmetric on $[0, 1]$ for all $x \in [0, +\infty)$.

This work is organized as follows. After this section, we give some preliminary lemmas. In Section 3, we give our main results Theorems 3.1 and 3.2. An example is also given to show our results. To the best of our knowledge, no paper has appeared in the literature which discusses the fourth-order integral boundary value problem for an increasing homeomorphism and homomorphism on time scales when nonlinearity in the dynamic equation may change sign.

In this paper, a symmetric positive solution x of (1.1) and (1.2) means a solution of (1.1) and (1.2) satisfying $x > 0$ and $x(t) = x(1 - t)$, $t \in [0, 1]$.

2. Preliminary lemmas

To prove the main results in this paper, we will make use of the following lemmas.

Lemma 2.1. *Assume* (H3) *holds. Then for any $y \in \mathcal{C}[0, 1]$, the BVP*

$$q(t)\phi(p(t)x^{\Delta\nabla}(t)) = y(t), \tag{2.1}$$

$$x(0) = x(1) = \int_0^1 g(s)x(s)\nabla s \tag{2.2}$$

has a unique solution x and x can be expressed in the form

$$x(t) = -\int_0^1 H(t, s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s, \tag{2.3}$$

where

$$H(t, s) = G(t, s) + \frac{1}{1 - \mu}\int_0^1 G(s, \tau)g(\tau)\nabla\tau, \tag{2.4}$$

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1; \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.5}$$

Proof. First suppose that $x \in \mathcal{C}[0, 1]$ is a solution of problem (2.1) and (2.2). It is easy to see by integration of both sides of (2.1) on $[0, t]$ that

$$x^{\Delta}(t) - x^{\Delta}(0) = \int_0^t \frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s.$$

Integrating again, we can get

$$x(t) = x(0) + x^{\Delta}(0)t + \int_0^t (t-s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s. \tag{2.6}$$

Letting $t = 1$ in (2.6), we find

$$x^{\Delta}(0) = -\int_0^1 (1-s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s. \tag{2.7}$$

Substituting $x(0) = \int_0^1 g(s)x(s)\nabla s$, we obtain

$$\begin{aligned} x(t) &= \int_0^1 g(s)x(s)\nabla s - \int_0^1 t(1-s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s + \int_0^t (t-s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s \\ &= -\int_0^1 G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s + \int_0^1 g(s)x(s)\nabla s, \end{aligned} \tag{2.8}$$

where

$$\int_0^1 g(s)x(s)\nabla s = \int_0^1 g(s)\left[-\int_0^1 G(s,\tau)\frac{1}{p(\tau)}\phi^{-1}\left(\frac{y(\tau)}{q(\tau)}\right)\nabla\tau + \int_0^1 g(\tau)x(\tau)\nabla\tau\right]\nabla s$$

and so

$$\int_0^1 g(s)x(s)\nabla s = \frac{-1}{1-\int_0^1 g(s)\nabla s}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\frac{1}{p(\tau)}\phi^{-1}\left(\frac{y(\tau)}{q(\tau)}\right)\nabla\tau\right]\nabla s. \tag{2.9}$$

Substituting (2.9) to (2.8), we have

$$\begin{aligned} x(t) &= -\int_0^1 G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s - \frac{1}{1-\mu}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\frac{1}{p(\tau)}\phi^{-1}\left(\frac{y(\tau)}{q(\tau)}\right)\nabla\tau\right]\nabla s \\ &= -\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s, \end{aligned} \tag{2.10}$$

where $H(t, s)$ is defined in (2.4).
Sufficiency let x be as in (2.10), then

$$\begin{aligned} x(t) &= -\int_0^t s(1-t)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s - \int_t^1 t(1-s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s \\ &\quad - \frac{1}{1-\mu}\int_0^1 g(s)\left[\int_0^1 G(s,\tau)\frac{1}{p(\tau)}\phi^{-1}\left(\frac{y(\tau)}{q(\tau)}\right)\nabla\tau\right]\nabla s. \end{aligned} \tag{2.11}$$

Taking the Δ derivative of (2.11), we get

$$\begin{aligned} x^{\Delta}(t) &= \int_0^t \frac{s}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s - (1-\sigma(t))\sigma(t)\frac{1}{p(\sigma(t))}\phi^{-1}\left(\frac{y(\sigma(t))}{q(\sigma(t))}\right) \\ &\quad + (1-\sigma(t))\sigma(t)\frac{1}{p(\sigma(t))}\phi^{-1}\left(\frac{y(\sigma(t))}{q(\sigma(t))}\right) - \int_t^1 \frac{1-s}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s \end{aligned}$$

and

$$x^{\Delta\nabla}(t) = \frac{t}{p(t)}\phi^{-1}\left(\frac{y(t)}{q(t)}\right) + \frac{1-t}{p(t)}\phi^{-1}\left(\frac{y(t)}{q(t)}\right) = \frac{1}{p(t)}\phi^{-1}\left(\frac{y(t)}{q(t)}\right),$$

and it is easy to verify that $x(0) = x(1) = \int_0^1 g(s)x(s)\nabla s$. The proof is complete. □

We will also assume the following condition.

(H4) $\int_0^1 G(s,\tau)g(\tau)\nabla\tau \neq 0$.

The functions G and H have the following properties.

Lemma 2.2. *Assume* (H3) *and* (H4) *hold, then we have*

$$H(t,s) > 0, \qquad G(t,s) > 0, \quad t,s \in [0,1].$$

Lemma 2.3. *For* $t,s \in [0,1]$,

$$G(t,s) \leq G(s,s), \quad G(1-t,1-s) = G(t,s).$$

Lemma 2.4. *Let* $\delta \in (0,1/2)$ *for all* $t \in J_\delta = [\delta, 1-\delta]$, $s \in [0,1]$, *then we have*

$$G(t,s) \geq \delta G(s,s).$$

Lemma 2.5. *If* (H3) *holds, then for all* $t,s \in [0,1]$ *we have*

$$H(1-t,1-s) = H(t,s).$$

Proof.

$$\begin{aligned} H(1-t,1-s) &= G(1-t,1-s) + \frac{1}{1-\mu}\int_0^1 G(1-s,\tau)g(\tau)\nabla\tau \\ &= G(t,s) + \frac{1}{1-\mu}\int_1^0 G(1-s,1-\tau)g(1-\tau)\nabla(1-\tau) \\ &= G(t,s) + \frac{1}{1-\mu}\int_0^1 G(s,\tau)g(\tau)\nabla\tau \\ &= H(t,s). \quad \square \end{aligned}$$

Lemma 2.6. *If* (H3) *holds, then for all* $t,s \in [0,1]$ *we have*

$$H(t,s) \leq H(s,s).$$

Lemma 2.7. *If* (H3) *holds, then for all* $t \in J_\delta = [\delta, 1-\delta]$, $s \in [0,1]$ *we have*

$$H(t,s) \geq \delta G(s,s),$$

where δ *is defined in Lemma 2.4.*

Lemma 2.8. *Assume* (H3) *holds. Then for any* $y \in \mathcal{C}[0,1]$, *the BVP*

$$y^{\Delta\nabla}(t) = \lambda f(t,x(t)), \tag{2.12}$$

$$y(0) = y(1) = \int_0^1 h(s)y(s)\nabla s \tag{2.13}$$

has a unique solution y *and* y *can be expressed in the form*

$$y(t) = -\lambda \int_0^1 H_1(t,s)f(s,x(s))\nabla s, \tag{2.14}$$

where

$$H_1(t,s) = G(t,s) + \frac{1}{1-\upsilon}\int_0^1 G(s,\upsilon)h(\upsilon)\nabla\upsilon.$$

Assume that x is a solution of problem (1.1) and (1.2). Then from Lemma 2.1, we get

$$x(t) = -\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{y(s)}{q(s)}\right)\nabla s.$$

From Lemma 2.8, we have

$$x(t) = -\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s.$$

Lemma 2.9. *If* (H3) *holds, then for all* $t, s \in [0, 1]$ *we have*

$$H_1(1-t, 1-s) = H_1(t, s).$$

Lemma 2.10. *If* (H3) *holds, then for all* $t, s \in [0, 1]$ *we have*

$$H_1(t, s) \leq H_1(s, s).$$

Lemma 2.11. *If* (H3) *holds, then for all* $t \in J_\delta = [\delta, 1-\delta]$, $s \in [0, 1]$ *we have*

$$H_1(t, s) \geq \delta G(s, s),$$

where δ *is defined in Lemma 2.4.*

To obtain a positive solution of BVP (1.1) and (1.2), the following fixed point theorem is essential.

Lemma 2.12 (*See [10]*)**.** *Let* K *be a cone in a Banach space* B. *Let* D *be an open bounded subset of* B *with* $D_K = D \cap K \neq \emptyset$ *and* $\overline{D_{\mathcal{K}}} \neq \mathcal{K}$. *Suppose that* $A : \overline{D_{\mathcal{K}}} \to \mathcal{K}$ *is a completely continuous map such that* $x \neq Ax$ *for* $x \in \partial D_K$. *Then the following results hold:*

(i) *If* $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, *then* $i(A, D_K, K) = 1$;
(ii) *If there exists* $x_0 \in K \setminus \{0\}$ *such that* $x \neq Ax + \lambda x_0$, *for all* $x \in \partial D_K$ *and all* $\lambda > 0$, *then* $i(A, D_K, K) = 0$;
(iii) *Let* U *be open in* E *such that* $\overline{U} \subset D_K$. *If* $i(A, D_K, K) = 1$ *and* $i(A, U_K, K) = 0$, *then* A *has a fixed point in* $D_K \setminus \overline{U_K}$. *The same result holds if* $i(A, D_K, K) = 0$ *and* $i(A, U_K, K) = 1$.

Consider the Banach space $\mathcal{C}[0, 1]$ equipped with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, and define the cone $\mathcal{K} \subset B$ by

$$\mathcal{K} = \{x : x \in \mathcal{C}[0, 1], x(t) \text{ is concave, nonnegative and symmetric on } [0, 1]\}.$$

We define

$$\varphi(t) = \min\{t, 1-t\}, \quad t \in (0, 1),$$

$$\gamma = \frac{\delta^2 \int_\delta^{1-\delta} G(s, s) \frac{1}{p(s)} \phi^{-1}\left[\delta \frac{1}{q(s)} \int_\delta^{1-\delta} G(\tau, \tau) \nabla\tau\right] \nabla s}{\int_0^1 H(s, s) \frac{1}{p(s)} \phi^{-1}\left[\frac{1}{q(s)} \int_0^1 H_1(\tau, \tau) \nabla\tau\right] \nabla s},$$

$$\gamma_1 = \frac{\delta \int_\delta^{1-\delta} G(s, s) \frac{1}{p(s)} \phi^{-1}\left[\delta \frac{1}{q(s)} \int_\delta^{1-\delta} G(\tau, \tau) \nabla\tau\right] \nabla s}{\int_0^1 H(s, s) \frac{1}{p(s)} \phi^{-1}\left[\frac{1}{q(s)} \int_0^1 H_1(\tau, \tau) \nabla\tau\right] \nabla s},$$

$$K_\rho = \{x(t) \in \mathcal{K} : \|x\| < \rho\},$$

$$K_\rho^* = \{x(t) \in \mathcal{K} : \rho\varphi(t) < x(t) < \rho\},$$

$$\Omega_\rho = \left\{x(t) \in \mathcal{K} : \min_{t\in[\delta,1-\delta]} x(t) < \gamma\rho\right\}.$$

Lemma 2.13. Ω_ρ *has the following properties:*

(a) Ω_ρ *is open relative to* $\mathcal{K}$.
(b) $\mathcal{K}_{\gamma\rho} \subset \Omega_\rho \subset \mathcal{K}_\rho$.
(c) $x \in \partial\Omega_\rho$ *if and only if* $\min_{t\in[\delta,1-\delta]} x(t) = \gamma\rho$.
(d) *If* $x \in \partial\Omega_\rho$, *then* $\gamma\rho \leq x(t) \leq \rho$ *for* $t \in [\delta, 1-\delta]$.

Now, for convenience, we introduce the following notations. Let

$$f_{\gamma\rho}^{\rho} = \min\left\{\frac{f(t, x)}{\phi(\rho)} : t \in [\delta, 1-\delta], x \in [\gamma\rho, \rho]\right\},$$

$$f_{\varphi(t)\rho}^{\rho} = \max\left\{\frac{f(t, x)}{\phi(\rho)} : t \in [0, 1], x \in [\rho\varphi(t), \rho]\right\},$$

$$m = \left[\int_0^1 H(s, s) \frac{1}{p(s)} \phi^{-1}\left[\frac{1}{q(s)} \int_0^1 H_1(\tau, \tau) \nabla\tau\right] \nabla s\right]^{-1},$$

$$M = \left[\delta \int_\delta^{1-\delta} G(s, s) \frac{1}{p(s)} \phi^{-1}\left[\delta \frac{1}{q(s)} \int_\delta^{1-\delta} G(\tau, \tau) \nabla\tau\right] \nabla s\right]^{-1}.$$

It is easy to see that $m > 0$, $M < \infty$, and $M\gamma = M\delta\gamma_1 = \delta^2 m < m$.

3. Existence of one symmetric positive solution

In this section, growth conditions are imposed on f which yield the existence of one symmetric positive solution of the eigenvalue problem (1.1) and (1.2) by using the fixed point index theorem.

Theorem 3.1. *Assume that* (H1)–(H4) *are satisfied. Furthermore, the following condition holds:*

(H5) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that*
 (1) $f(t, x) > 0,\ t \in [0, 1], x \in [\rho_1\varphi(t), \infty)$;
 (2) $\frac{\phi(M\gamma)}{f_{\gamma\rho_2}^{\rho_2}} \le \lambda \le \frac{\phi(m)}{f_{\varphi(t)\rho_1}^{\rho_1}}$.

Then BVP (1.1) *and* (1.2) *has one symmetric positive solution in* $\mathcal{K}$.

Proof. Assume that (H5) holds.

$$f^*(t, x) = \begin{cases} f(t, x), & x \ge \rho_1\varphi(t); \\ f(t, \rho_1\varphi(t)), & 0 \le x < \rho_1\varphi(t). \end{cases}$$

It is easy to check that $f^*(t, x) \in \mathcal{C}([0, 1] \times [0, +\infty), (0, +\infty))$ and since $f(t, x)$ is symmetric on the interval $[0, 1]$, $f^*(t, x)$ is also symmetric on the interval $[0, 1]$.

Now, we consider the following modified problem of (3.1) and (3.2)

$$(q(t)\phi(p(t)x^{\Delta\nabla}))^{\Delta\nabla} = \lambda f^*(t, x(t)), \quad t \in (0, 1), \tag{3.1}$$

$$x(0) = x(1) = \int_0^1 g(s)x(s)\nabla s,$$

$$q(0)\phi(p(0)x^{\Delta\nabla}(0)) = q(1)\phi(p(1)x^{\Delta\nabla}(1)) = \int_0^1 h(s)q(s)\phi(p(s)x^{\Delta\nabla}(s))\nabla s. \tag{3.2}$$

It is easy to see that the BVP (3.1) and (3.2) has a solution $x = x(t)$ if and only if x is a fixed point of the operator equation

$$Ax(t) = -\int_0^1 H(t, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla s. \tag{3.3}$$

For all $x \in \mathcal{K}$, we have by

$$\begin{aligned}(Ax)^{\Delta\nabla}(t) &= \phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s, \tau)f^*(\tau, x(\tau))\nabla\tau\right) \\ &\le 0,\end{aligned}$$

which implies Ax is concave on $[0, 1]$.

On the other hand, by (3.3) we have

$$(Ax)(0) = Ax(1) = \int_0^1 H(0, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla s \ge 0.$$

It follows that $Ax(t) \ge 0$ for $t \in [0, 1]$. Noticing that $p(t),\ q(t)$ are symmetric on $[0, 1]$, $x(t)$ is symmetric on $[0, 1]$, and $f^*(t, x)$ is symmetric on $[0, 1]$ we have

$$\begin{aligned}Ax(1-t) &= -\int_0^1 H(1-t, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla s \\ &= -\int_1^0 H(1-t, 1-s)\frac{1}{p(1-s)}\phi^{-1}\left(-\lambda\frac{1}{q(1-s)}\int_0^1 H_1(1-s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla(1-s) \\ &= -\int_0^1 H(t, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(1-s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla s \\ &= -\int_0^1 H(t, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_1^0 H_1(1-s, 1-\tau)f^*(1-\tau, x(1-\tau))\nabla(1-\tau)\right)\nabla s \\ &= -\int_0^1 H(t, s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s, \tau)f^*(\tau, x(\tau))\nabla\tau\right)\nabla s \\ &= (Ax)(t),\end{aligned}$$

i.e. $(Ax)(1-t)=(Ax)(t)$, $t\in[0,1]$. Therefore, $(Ax)(t)$ is symmetric on $[0,1]$. By applying the Arzela–Ascoli Theorem on time scales, we can obtain that $A(K)$ is relatively compact. In view of the Lebesgue Convergence Theorem on time scales, it is obvious that A is continuous. Hence, $A:\mathcal{K}\mapsto\mathcal{K}$ is a completely continuous operator.

From the condition (H5) (2), we have $\frac{\phi(M\gamma)}{f^{*\rho_2}_{\gamma\rho_2}}\le\lambda\le\frac{\phi(m)}{f^{\rho_1}_{\varphi(t)\rho_1}}$.

Firstly, we show that $i(A,\mathcal{K}^*_{\rho_1},\mathcal{K})=1$.

In fact by $\lambda f^{*}_{\varphi(t)\rho_1}{}^{\rho_1}\le\phi(m)$ and $x\ne Ax$, for $x\in\partial\mathcal{K}^*_{\rho_1}$, we have for $x\in\partial\mathcal{K}^*_{\rho_1}$,

$$-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x(\tau))\nabla\tau\ge-\phi(m)\phi(\rho_1)\frac{1}{q(s)}\int_0^1 H_1(\tau,\tau)\nabla\tau,$$

so that

$$\varphi(s)=\phi^{-1}\left[-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x(\tau))\nabla\tau\right]\ge-m\rho_1\phi^{-1}\left[\frac{1}{q(s)}\int_0^1 H_1(\tau,\tau)\nabla\tau\right].$$

Therefore we get,

$$\begin{aligned}Ax(t)&=-\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x(\tau))\nabla\tau\right)\nabla s\\&\le m\rho_1\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 H_1(\tau,\tau)\nabla\tau\right)\nabla s\\&\le m\rho_1\int_0^1 H(s,s)\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 H_1(\tau,\tau)\nabla\tau\right)\nabla s\\&=\rho_1=\|x\|.\end{aligned}$$

So,

$$\|Ax\|\le\|x\|,\quad\forall x\in\partial\mathcal{K}^*_{\rho_1}.$$

By Lemma 2.12(i), we have $i(A,\mathcal{K}^*_{\rho_1},\mathcal{K})=1$.

Secondly, we show that $i(A,\Omega_{\rho_2},\mathcal{K})=0$. Let $e(t)\equiv1$, for $t\in[0,1]$, then $e\in\partial\mathcal{K}_1$. We claim that $x\ne Ax+\widetilde{\lambda}e$ for $x\in\partial\Omega_{\rho_2}$, and $\widetilde{\lambda}>0$. In fact, if not, there exist $x_0\in\partial\Omega_{\rho_2}$ and $\widetilde{\lambda_0}>0$ such that $x_0=Ax_0+\widetilde{\lambda}_0e$.

By (H5), $\lambda f^{\rho_2}_{\gamma\rho_2}\ge\phi(M\gamma)$, and $x\ne Ax$ for $x\in\partial\Omega_{\rho_2}$, we have for $t\in[\delta,1-\delta]$,

$$\begin{aligned}-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x_0(\tau))\nabla\tau&\le-\delta\lambda\frac{1}{q(s)}\int_\delta^{1-\delta}G(\tau,\tau)f^*(\tau,x_0(\tau))\nabla\tau\\&\le-\phi(\rho_2)\phi(M\gamma)\delta\frac{1}{q(s)}\int_\delta^{1-\delta}G(\tau,\tau)\nabla\tau,\end{aligned}$$

so that

$$\begin{aligned}\varphi(s)&=\phi^{-1}\left[-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x_0(\tau))\nabla\tau\right]\\&\le-\rho_2M\gamma\phi^{-1}\left[\delta\frac{1}{q(s)}\int_\delta^{1-\delta}G(\tau,\tau)\nabla\tau\right].\end{aligned}$$

Therefore,

$$\begin{aligned}x_0(t)&=Ax_0(t)+\widetilde{\lambda}_oe(t)\\&=-\int_0^1 H(t,s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x_0(\tau))\nabla\tau\right)\nabla s+\widetilde{\lambda_0}\\&\ge-\delta\int_\delta^{1-\delta}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(-\lambda\frac{1}{q(s)}\int_0^1 H_1(s,\tau)f^*(\tau,x_0(\tau))\nabla\tau\right)\nabla s+\widetilde{\lambda_0}\\&\ge\rho_2\delta M\gamma\int_\delta^{1-\delta}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\delta\frac{1}{q(s)}\int_\delta^{1-\delta}G(\tau,\tau)\nabla\tau\right)\nabla s+\widetilde{\lambda_0}\\&=\gamma\rho_2+\widetilde{\lambda_0}.\end{aligned}$$

This implies that $\gamma\rho_2\ge\gamma\rho_2+\widetilde{\lambda_0}$, a contradiction. Hence by Lemma 2.12(ii), it follows that $i(A,\Omega_{\rho_2},\mathcal{K})=0$.

By Lemma 2.13(b) and $\rho_1 < \gamma\rho_2$, we have $\overline{\mathcal{K}}_{\rho_1} \subset \mathcal{K}_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.12(iii) that A has a fixed point x_1 in $\Omega_{\rho_2} \setminus \overline{\mathcal{K}^*}_{\rho_1}$, we note that $f^*(t,x) = f(t,x)$, if $x \geq \rho_1\varphi(t)$. Thus we can get that problem (1.1) and (1.2) has a symmetric positive solution. The proof is complete. □

Theorem 3.2. *Suppose that* (H1)–(H4) *hold. Furthermore, the following condition holds:*

(H6) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that*
  (1) $f(t,x) > 0$, $t \in [0,1]$, $x \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;
  (2) $\frac{\phi(M\gamma)}{f_{\gamma\rho_1}^{\rho_1}} \leq \lambda \leq \frac{\phi(m)}{f_{\varphi(t)\rho_2}^{\rho_2}}$.

Then BVP (1.1) and (1.2) has one symmetric positive solution in $\mathcal{K}$.

Proof. The proof holds similar to that in (H5). So we omit it here. □

Example 3.1. Let $\mathbb{T} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and $f(t,0) \equiv 0$. Consider the following boundary value problem on $\mathbb{T}$

$$(\phi(x^{\Delta\nabla}))^{\Delta\nabla} = \lambda f(t,x(t)), \quad t \in (0,1), \tag{3.4}$$

$$x(0) = x(1) = \int_0^1 \frac{1}{2}x(s)\nabla s,$$

$$\phi(x^{\Delta\nabla}(0)) = \phi(x^{\Delta\nabla}(1)) = 0, \tag{3.5}$$

where $\phi(x) = x$,

$$f(t,x) = \begin{cases} t(1-t)\left(x(t) - \dfrac{\varphi(t)}{2}\right)^5, & (t,x) \in [0,1] \times (0,8]; \\ t(1-t)\left(8 - \dfrac{\varphi(t)}{2}\right)^5, & (t,x) \in [0,1] \times (8,+\infty). \end{cases}$$

It is obvious that $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous and f is symmetric on the interval [0, 1]. In this case $p(t) = 1$, $q(t) = 1$, $g(t) = 1/2$ and $h(t) = 0$, $\mu = \frac{1}{2}$, $\upsilon = 0$, it follows from a direct calculation that

$$H(t,s) = G(t,s) + \int_0^1 G(s,\tau)\nabla\tau, \qquad H_1(t,s) = G(t,s),$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1; \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

$$\begin{aligned} m &= \left[\int_0^1 H(s,s)\frac{1}{p(s)}\phi^{-1}\left[\frac{1}{q(s)}\int_0^1 H_1(\tau,\tau)\nabla\tau\right]\nabla s\right]^{-1} \\ &= \frac{3}{2}\left[\int_0^1 s(1-s)\left[\int_0^1 \tau(1-\tau)\nabla\tau\right]\nabla s\right]^{-1} \\ &\simeq 25.8. \end{aligned}$$

Choose $\delta = \frac{1}{3} \in \left(0, \frac{1}{2}\right)$, then

$$\begin{aligned} M &= \left[\delta\int_\delta^{1-\delta} G(s,s)\frac{1}{p(s)}\phi^{-1}\left[\delta\frac{1}{q(s)}\int_\delta^{1-\delta} G(\tau,\tau)\nabla\tau\right]\nabla s\right]^{-1} \\ &= \left[\frac{1}{3}\int_{\frac{1}{3}}^{\frac{2}{3}} s(1-s)\left[\frac{1}{3}\int_{\frac{1}{3}}^{\frac{2}{3}} \tau(1-\tau)\nabla\tau\right]\nabla s\right]^{-1} \\ &= 1640.25 \end{aligned}$$

$$\gamma = \frac{\delta\frac{1}{M}}{\frac{1}{m}} = \frac{8}{507}.$$

Choose $\rho_1 = 1$, $\rho_2 = 507$, it is easy to check that $1 = \rho_1 < \gamma\rho_2 = \frac{8}{507} \times 507 = 8$.

$$f(t,x) > 0, \quad t \in [0,1], \qquad u \in [\varphi(t),+\infty).$$

$$f^{\rho_1}_{\varphi(t)\rho_1} = \max\left\{\frac{t(1-t)\left(x(t)-\frac{\varphi(t)}{2}\right)^5}{1} : t\in[0,1], x\in[\varphi(t),1]\right\}$$
$$= \frac{\frac{1}{2}\left(1-\frac{1}{2}\right)1^5}{1} = \frac{1}{4}$$
$$f^{\rho_2}_{\gamma\rho_2} = \min\left\{\frac{t(1-t)\left(8-\frac{\varphi(t)}{2}\right)^5}{507} : t\in\left[\frac{1}{3},\frac{2}{3}\right], x\in[8,507]\right\}$$
$$= \frac{\frac{1}{3}\left(1-\frac{1}{3}\right)\left(8-\frac{1}{2}\right)^5}{507} \simeq 10.4.$$

So

$$\frac{\phi(M\gamma)}{f^{\rho_2}_{\gamma\rho_2}} \simeq 0.82, \qquad \frac{\phi(m)}{f^{\rho_1}_{\varphi(t)\rho_1}} = 103.2.$$

Thus, if $0.82 \leq \lambda \leq 103.2$, then the BVP (3.4) and (3.5) has one symmetric positive solution.

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References

[1] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, in: Nonlinear Dyn. Quantum Dyn. Sys., (Gaussing, 1990), in: Math. Res., vol. 59, Akademie Verlag, Berlin, 1990, pp. 9–20.
[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An introduction with Applications, Birkhäuser, Boston, 2001.
[3] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[4] Junfang Zhao, Chunmei Miao, Weigao Ge, Jiani Zhang, Multiple symmetric positive solutions to a new kind of four point boundary value problem, Nonlinear Analysis 71 (2009) 9–18.
[5] Gennaro Infante, Eigenvalues and positive solutions of ODES involving integral boundary conditions, Computers and Mathematics with Applications 48 (2004) 517–527.
[6] Q. Yao, Existence and iteration of n symmetric positive solutions for a singular two-point boundary value problem, Computers and Mathematics with Applications 47 (2004) 1195–1200.
[7] R.I. Avery, J. Henderson, Three symmetric positive solutions for a second-order boundary value problem, Applied Mathematics Letters 13 (2000) 1–7.
[8] Xuemei Zhang, Meiqiang Feng, Weigao Ge, Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions, Journal of Computational and Applied Mathematics 222 (2008) 561–573.
[9] Huili Ma, Symmetric positive solutions for nonlocal boundary value problems of fourth order, Nonlinear Analysis 68 (2008) 645–651.
[10] D. Guo, V. Lashmikanthan, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.